

# Uplink performance of multi-antenna cellular networks with co-operative base stations and user-centric clustering

Siddhartan Govindasamy, Itsik Bergel

## Abstract

We consider a user-centric co-operative cellular network, where base stations close to a mobile co-operate to detect its signal using a (joint) linear minimum-mean-square-error receiver. The base stations, which have multiple antennas, and mobiles are modeled as independent Poisson Point Processes to avoid dependence on specific node locations. Combining stochastic geometry and infinite random matrix theory, we derive a simple expression for the spectral efficiency of this complex system as the number of antennas grows large. This expression is verified by Monte Carlo simulations which support its utility for even a moderate number of antennas. This result reveals the influence of tangible system parameters such as mobile and base-station densities, number of antennas per base station, and number of co-operating base stations on achievable spectral efficiencies. For instance, we find that for a given base station density and a constraint on the total number of co-operating antennas, all co-operating antennas should be located at a single base station. On the other hand, in our asymptotic regime, for the same number of co-operating antennas, if the network is limited by the area density of antennas, then the number of co-operating base stations should be increased with fewer antennas per base station.

## Index Terms

Cellular Networks, MIMO, Antenna Arrays, Stochastic Geometry, Poisson Point Process

## I. INTRODUCTION

Co-operative processing has attracted much attention in the analysis of cellular networks, both for transmission and for reception. Such systems have been proposed in the context of cloud-radio access networks (CRAN) (see e.g. [2]) and distributed massive multiple-input multiple-output (MIMO) systems (see e.g. [3] – [6]), and can offer an increase in spatial diversity as signals from the antennas on multiple base stations can be used for detection. Most approaches analyzed in the literature use static clustering of base-stations (BSs) for joint processing, which creates ‘super-cells’ in which the clustered BSs function as a single BS with greater capabilities. While this approach provides increased spatial diversity, it still has a problem with users that are located at the edges of the new ‘super-cells’. A more flexible approach is to group the clusters dynamically, such that the signals to/from the  $K$  closest base stations to a given mobile user are jointly processed. Such an approach, which is a form of user-centric clustering, can completely resolve the cell-edge issue, at the cost of greater complexity in coordination.

This work aims at analyzing the uplink of a cellular network with user-centric clustering and multi-antenna base stations, where the mobiles and base stations are assumed to be distributed on a plane according to Poisson Point Processes (PPPs). We assume that the  $K$  base stations closest to a given mobile cooperate to decode its signal using the linear Minimum-Mean-Square Error (MMSE) estimator, which is the optimal linear estimator to maximize the Signal-to-Interference Ratio (SIR) of the post-processed signal, under the assumption that the thermal noise is negligible. The performance of this system depends on the number of antennas per base station  $L$ , and the spatial density of base stations, and mobiles. The resulting asymptotic expressions for the achievable spectral efficiency are simple, and reveal the dependencies between user densities and base station densities, the number of co-operating base stations and the number of antennas per base station, which enables designers to understand the tradeoffs between increasing the number of antennas per base station, and the number of co-operating base stations to handle increasing densities of mobiles. Due to the optimality of MMSE estimator, this result also provides an upper bound on the performance that can be expected from all linear receivers.

Modeling the spatial distribution of base stations and mobiles as PPPs is a powerful technique to analyze wireless cellular networks and has been used in a number of works such as [7]–[11], over a variety of different system models. Such models “lead to remarkably precise characterizations” [7], and as such are very useful in the analysis of wireless networks.

As was pointed out in [12], most works on user-centric base-station clustering have relied on numerical simulations. Analytical results on such systems are limited in the literature due to a number of technical challenges as described later in this section. Understanding the potential performance benefits of such a system is important as base-station clustering can provide very high data rates, but the cost associated with jointly processing signals received on multiple, spatially distributed base stations is

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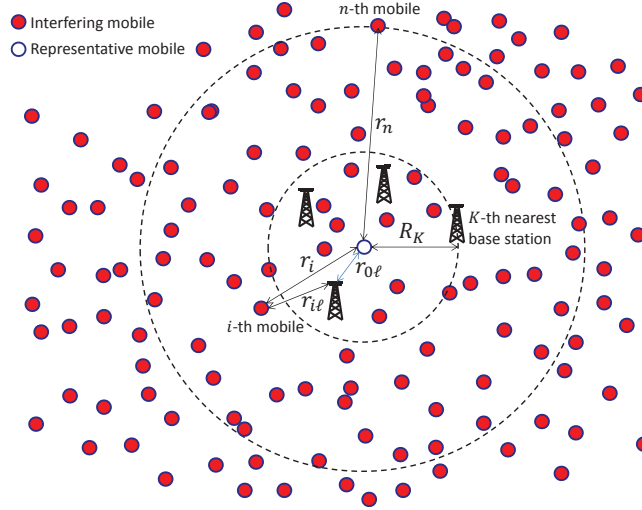


Fig. 1. Illustration of network with base-station cooperation.

potentially very high. As such, analytical results which indicate the dependencies of different system parameters on achievable data rates are very useful.

While in the context of massive MIMO systems, it was shown that the matched filter (MF) receiver is asymptotically optimal [13], there is a wide range of realistic system parameters for which the MMSE receiver performs significantly better than the MF receiver [14]. Therefore, for practical (not so) massive MIMO systems, analyzing the MMSE receiver is very useful. Additionally, we assume that channel information required to construct the MMSE receiver is known accurately in this work. As such these results are achievable as long as the number of antennas is not extremely large (where channel estimation and pilot contamination become significant challenges), and moreover, this result is useful as an upper bound on the performance in situations where channel estimation errors are significant.

Analyzing the uplink of co-operative base station systems is complicated by the fact that the signals from a given mobile experience different path losses to the arrays of the co-operating base stations. While a number of works have considered the multi-antenna MMSE receiver in spatially distributed ad-hoc and non-cooperative cellular networks (e.g., [15] – [17]), these works rely on the fact that channel matrices for their respective models can be factored into the product of two matrices, one representing fast fading, and a diagonal matrix containing the square-root of path-losses. Such a factorization is not possible when the antennas receiving the signal from a given mobile experience different path-losses, as would be the case for a co-operative system. We address this complexity using an asymptotic analysis as the number of antennas per base station grows large.

A number of works have analyzed the performance of base-station clustering systems in networks where the spatial distribution of users and base stations are considered. The majority of these works focus on the downlink, and include systems with both fixed clusters, (e.g., [10]) and user-centric clustering. As observed in [12], most works on user-centric-clustering have used numerical simulations, e.g., [18] which considered zero-forcing beamformers in the uplink, [19] which considered the downlink using sparse beamforming, and [20] which considered a number of precoding strategies for the downlink. In [21], and [22], the downlink of user-centric base station co-operation systems is considered analytically. Very recently, [12] analyzed both the uplink and downlink of multi-antenna distributed processing systems using zero-forcing beamformers, and user-centric clustering. In [12], the significant complexities in analyzing the uplink of multi-antenna base-station cooperation systems are handled using a number of approximations, e.g., on the distribution of the interference. In this work on the other hand, we use an asymptotic analysis with the optimal linear receiver, and derive a compact expression for the uplink spectral efficiency on a representative multi-antenna link.

## II. SYSTEM MODEL

We consider a network of base stations and mobiles, each distributed according to an independent, homogenous Poisson Point Process (HPPP) on the plane with area densities of  $\lambda_b$  base stations per unit area and  $\lambda$  mobiles per unit area, respectively. In this work we analyze the performance of the uplink in a cellular cloud-radio network where the transmission from each mobile is decoded from the signals received by its  $K$  closest BSs in Euclidian distance.

Our approach is to analyze the performance of a typical link, between a representative mobile transmitter located at the origin of a planar network, and an MMSE receiver applied to signals received on the antennas of the  $K$  base-stations closest to the origin. Due to the homogeneity of the network, the statistical properties of links between other mobiles and their respective MMSE receivers are statistically equivalent.

The representative (or test) mobile shall be labeled as mobile-0, whereas all other mobiles, will be co-channel interferers to the test mobile. For simplicity of notation, we order both BS and mobiles according to their distance from the test mobile (although a different (random) ordering will be used in portions of the proof). Let  $r_i$  denote the distance from the representative mobile to the  $i$ -th mobile and  $r_{i,j}$  denote the distance between the  $i$ -th mobile and the  $j$ -th BS (thus, our ordering results with  $r_1 \leq r_2 \leq r_3 \dots$  and  $r_{0,1} \leq r_{0,2} \leq r_{0,3} \dots$ ). Fig. 1 illustrates this network, where we have only shown 4 of the base stations. We assume that each base station has  $L$  antennas.

Ignoring the effects of thermal noise, at a given sampling time, stacking the received signals at the  $L$  antennas of the  $K$  co-operating base stations into a vector  $\mathbf{y} \in \mathbb{C}^{LK \times 1}$ , we have

$$\mathbf{y} = \mathbf{h}_0 x + \sum_{i=1}^{\infty} \mathbf{h}_i x_i \quad (1)$$

where  $x_i$  is the zero-mean, unit variance transmitted symbol from the  $i$ -th mobile and

$$\mathbf{h}_i = \left( g_{i,1} \cdot r_{i,1}^{-\alpha/2}, \dots, g_{i,L} \cdot r_{i,1}^{-\alpha/2}, g_{i,L+1} \cdot r_{i,2}^{-\alpha/2}, \dots, g_{i,2L} \cdot r_{i,2}^{-\alpha/2}, \dots, g_{i,LK} \cdot r_{i,K}^{-\alpha/2} \right)^T. \quad (2)$$

Here  $\alpha > 2$  is the path-loss exponent and  $g_{i,k}$  are independent and identically distributed (i.i.d.)  $\mathcal{CN}(0, 1)$  random variables that represent fast fading, where the notation  $\mathcal{CN}(0, 1)$  indicates a zero-mean, circularly symmetric, unit-variance complex Gaussian random variable. Thus,  $\mathbf{h}_i$  is a vector constructed by stacking the channel vectors between the  $i$ -th mobile and each of the  $K$  base stations co-operating to detect the signal from the representative mobile.

To avoid the use of matrices with infinite dimension, we first consider the subsystem with only the first  $n$  mobiles. The received signal in this subsystem is

$$\mathbf{y}[n] = \mathbf{h}_0 x + \sum_{i=1}^n \mathbf{h}_i x_i. \quad (3)$$

Considering this  $n$ -mobiles subsystem, the symbol transmitted by the test mobile is estimated using a linear Minimum-Mean-Square Error (MMSE) estimator:

$$\mathbf{c}[n] = \beta \left( \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-1} \mathbf{h}_0, \quad (4)$$

where  $\mathbf{H}[n] = [\mathbf{h}_1 | \dots | \mathbf{h}_n]$ , and  $\beta$  is some scale factor that does not impact the SIR as it scales the interference and signal by the same value. Note that the expression for the MMSE estimator above does not include thermal noise as our focus is on the interference-limited regime. Thus, the SIR of the post-processed uplink signal from the representative mobile is given by

$$\text{SIR}[n] = \mathbf{h}_0^\dagger \left( \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-1} \mathbf{h}_0. \quad (5)$$

Note that the matrix  $\mathbf{H}[n] \mathbf{H}^\dagger[n]$  in (5) is invertible with high probability whenever  $n > KL$ . Thus, in the following, we limit the discussion to  $n > KL$ . For notational convenience, we shall define  $N = KL$ , which is the total number of antennas used to detect the signal from the representative mobile.

The matrix  $\left( \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-1}$  is positive definite and monotonically decreasing with  $n$  (in the positive definite sense). Hence, the limit  $\lim_{n \rightarrow \infty} \left( \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-1}$  exists, and we can define the estimator and the SIR of the original system with all the mobiles as:

$$\mathbf{c} = \lim_{n \rightarrow \infty} \mathbf{c}[n] \quad (6)$$

and

$$\text{SIR} = \lim_{n \rightarrow \infty} \text{SIR}[n] \quad (7)$$

respectively. Note that the SIR is a random variable that depends on the locations of the mobiles and on the channel fading. Moreover, by assuming that all mobiles in the network use Gaussian codebooks, we define the spectral efficiency of the system as

$$\eta = \log_2(1 + \text{SIR}) \quad (8)$$

In this work we study the behavior of the SIR and spectral efficiency in the asymptotic regime as  $L$  goes to infinity. This approach provides insight into the scaling behavior of these quantities as  $L$  increases, but more importantly, it provides approximations for the spectral efficiency and SIR when  $L$  is fixed, but large.

### III. MAIN RESULTS

#### A. Spectral Efficiency of a Representative Link

Considering the system model above, the following theorem characterizes the spectral efficiency in the uplink of the test mobile.

*Theorem 1:* Let  $\mathbf{P}_K = L \sum_{k=1}^K r_{0,k}^{-\alpha}$  denote the total average power received from the test mobile by all co-operating BSs. Conditioning on  $\mathbf{P}_K$  and assuming that Gaussian codebooks are used by each mobile and neglecting thermal noise, the spectral efficiency in the uplink of the test mobile, satisfies the following limiting behavior in probability:

$$\lim_{L \rightarrow \infty} |\eta - \eta_{\text{asym}}| = 0 \quad (9)$$

where the asymptotic spectral efficiency is defined as follows

$$\eta_{\text{asym}} = \log_2 \left( 1 + \mathbf{P}_K \left[ \frac{\alpha}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}} (KL)^{\alpha/2-1} \right). \quad (10)$$

*Proof:* The proof of this theorem relies on the following lemma on the convergence of a scaled version of the SIR.

*Lemma 1:* The SIR in the uplink of the test mobile, satisfies the following limiting behavior in probability:

$$\lim_{L \rightarrow \infty} \frac{\text{SIR}}{(KL)^{\alpha/2-1} \mathbf{P}_K} = \left[ \frac{\alpha}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}} \quad (11)$$

*Proof:* See Appendix A

We can use this lemma to show that the spectral efficiency  $\eta$ , approaches the asymptotic spectral efficiency  $\eta_{\text{asym}}$ , in probability as follows.

$$\begin{aligned} |\eta - \eta_{\text{asym}}| &= \left| \log_2(1 + \text{SIR}) - \log_2 \left( 1 + \left[ \frac{\alpha}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}} (KL)^{\alpha/2-1} \mathbf{P}_K \right) \right| \\ &= \left| \log_2 \left( \frac{\frac{1}{(KL)^{\alpha/2-1} \mathbf{P}_K} + \frac{\text{SIR}}{(KL)^{\alpha/2-1} \mathbf{P}_K}}{\frac{1}{(KL)^{\alpha/2-1} \mathbf{P}_K} + \left[ \frac{\alpha}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}}} \right) \right| \end{aligned}$$

Since  $\mathbf{P}_K$  is monotonically increasing with  $L$  and  $\alpha > 2$ , by Lemma 1 and the continuous mapping theorem, the left hand side terms in both the numerator and the denominator in the log term above vanish in the limit, and we have (9) in probability. ■

Thus, for large enough  $L$ , the spectral efficiency is well approximated as

$$\eta \approx \eta_{\text{asym}} = \log_2(1 + \text{SIR}_{\text{asym}}), \quad (12)$$

where

$$\text{SIR}_{\text{asym}} = \mathbf{P}_K \left[ \frac{\alpha}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}} (KL)^{\alpha/2-1}. \quad (13)$$

It is convenient to note that  $\mathbf{P}_K$  is the average power received by the  $K$  co-operating BSs. Thus, we define the asymptotic interference power:

$$\sigma^2 = \left[ \frac{\alpha}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{-\frac{\alpha}{2}} (KL)^{1-\alpha/2} \quad (14)$$

and we have  $\text{SIR}_{\text{asym}} = \mathbf{P}_K / \sigma^2$ . Equation (14) shows that the the number of co-operating BSs,  $K$ , and the number of antennas per BS have the same effect on the limit interference power, i.e., the limit interference power depends only on the total number of co-operating antennas, for a particular  $\mathbf{P}_K$ . On the other hand, the average received power,  $\mathbf{P}_K$ , has a completely different dependence on  $L$  and  $K$ .

To gain a better understanding, we consider a scaling of the network by a factor of  $\sqrt{\lambda_b}$ , with the origin as a reference point, which results in a unit-density network. Denoting the distance of the  $k$ -th base station from the origin in the scaled network by  $\hat{r}_{0,k} = r_{0,k} \sqrt{\lambda_b}$ , we define:

$$\hat{\mathbf{S}}_K = \frac{\mathbf{P}_K}{\lambda_b^{\alpha/2} KL} = \frac{1}{K} \sum_{k=1}^K \hat{r}_{0,k}^{-\alpha}. \quad (15)$$

Note that  $\hat{\mathbf{S}}_K$  represents the average power from the  $K$  nearest transmitters in a HPPP with a unit density. Hence, the asymptotic SIR can be written as:

$$\text{SIR}_{\text{asym}} = \left[ \frac{\alpha \lambda_b}{2\pi^2\lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}} (KL)^{\alpha/2} \hat{\mathbf{S}}_K \quad (16)$$

in which  $\hat{\mathbf{S}}_K$  is a normalizing random variable, which depends only on  $K$ , and the distances from the origin of the  $K$  closest points in a HPPP of unit density. Note that as expected for an interference-limited system, it is the relative density of base stations to mobiles, rather than the specific values of each that matters.

Equation (16) also reveals the relation between the number of antennas per BS,  $L$ , and the mobile density  $\lambda$ . For a given  $K$ , if we wish to increase the density of active mobiles while keeping the same level of performance per mobile, then the number of antennas per BS should scale linearly with the  $\lambda$ . This result is quite expected, as a similar result was obtained for cellular massive MIMO without BS cooperation [17].

Additionally, observe that  $\sum_{k=1}^K \hat{r}_{0,k}^{-\alpha}$  is monotonically increasing with  $K$ . Thus, (12), (15) and (16) indicate that we can support an increasing density of mobiles if we ensure the quantity  $\frac{\lambda_b L K^{1-\frac{2}{\alpha}}}{\lambda}$  is kept constant.

### B. Network Optimization

The results above give a way to evaluate the network performance and understand the role of each network parameter on the SIR and spectral efficiency in the system. In Subsection III-C below we give a formula for the CDF of the spectral efficiency in this system. Before doing so, we first consider the question of network optimization, i.e., the optimal choice of the network parameters given network constraints. This optimization sheds some light on the network design problem, and also helps in better understanding the effect of different network parameters on the SIR and spectral efficiency.

1) *Optimization given the number of co-operating antennas:* One simple measure of the network complexity is the number of co-operating antennas. For example, in CRANs the received signals from many base stations are transferred to a central processing center. With  $K$  co-operating base stations with  $L$  antennas each,  $KL$  signals need to be transferred to the central processing unit. If  $KL$  is large, this results in significant overhead on the infrastructure. Thus, the number of processed signals,  $KL$ , characterizes the complexity in the infrastructure required to share signals from multiple base stations.

Trying to optimize (16) with respect to  $K$  and  $L$  for a given complexity  $N = KL$ , reveals that having a larger  $L$  is obviously better. This is because most of the formula depends directly on  $N$ , except for  $\hat{\mathbf{S}}_K$  which is monotonically decreasing with  $K$  (see (15)). Thus, (16) shows that if we are only limited by the number of processed antennas, it is better to have more antennas in each BS, and fewer co-operating base stations.

2) *Optimization with limited antenna density:* But, the optimization of the previous subsection only takes into account the complexity of sharing signals from the  $N$  antennas. As a result, the preferred solution requires a large number of antennas per base station and hence, a large per-base-station hardware cost. To account for this hardware, we add a constraint on the density of antennas in the network. The antenna density is given by  $\lambda_a = \lambda_b L$  antennas per unit area.

In a network with an antenna density of  $\lambda_a$ , we can write the asymptotic SIR, (16), as:

$$\text{SIR}_{\text{asym}} = \left[ \frac{\alpha}{2\pi^2 \lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}} \lambda_a^{\alpha/2} K^{\alpha/2-1} (K \hat{\mathbf{S}}_K). \quad (17)$$

Noting that  $\alpha > 2$  and that  $K \hat{\mathbf{S}}_K$  is monotonically increasing with  $K$ , we conclude that the performance is monotonically increasing with  $K$ . Thus, we prefer to have more co-operating BSs with less antenna per BS (which is the opposite from the conclusion in the previous subsection).

To understand the difference between the conclusions of this subsection and the previous one, we note that in both cases, the asymptotic interference power,  $\sigma^2$  depends only in the total number of co-operating antennas. Thus, the differences result from the optimization of the average received power in each scenario ( $\mathbf{P}_K$ ). As  $\alpha > 2$ , the most important factor for energy collection is the probability of having a BS which is very close to the test mobile. Thus, if the BSs density ( $\lambda_b$ ) is fixed, increasing the number of antennas per BS increases the received energy from the nearest BS. On the other hand, when the BS density varies, it is best to distribute the BSs as much as possible in order to increase the probability of being close to the test mobile. Hence, in such case it is better to increase  $\lambda_b$  and  $K$  while decreasing  $L$  (provided, of course the asymptotic expressions still hold).

### C. Distribution of asymptotic spectral efficiency

Theorem 1 characterizes the asymptotic spectral efficiency of a representative link conditioned on  $\mathbf{P}_K$ . If we remove this conditioning, the asymptotic spectral efficiency,  $\eta_{\text{asym}}$  becomes a random variable. The distribution of  $\eta_{\text{asym}}$ , is quite complicated, due to its dependence on  $\mathbf{P}_K$ . The distribution of  $\mathbf{P}_K$  is characterized in the following lemma.

*Lemma 2:* The CDF of  $\mathbf{P}_K/L = \sum_{i=1}^K r_{0,i}^{-\alpha}$  is given by

$$\begin{aligned} F(x) &\triangleq P\left(\sum_{i=1}^K r_{0,i}^{-\alpha} \leq x\right) \\ &= e^{-\lambda_b \pi \left(\frac{K}{x}\right)^{2/\alpha}} \sum_{\ell=0}^K \frac{\left(\lambda_b \pi \left(\frac{K}{x}\right)^{2/\alpha}\right)^\ell}{\ell!} + \frac{1}{\pi} \sum_{\ell=1}^K \binom{K}{\ell} \left(-\Gamma\left(1 - \frac{2}{\alpha}\right)\right)^\ell \sin\left(\frac{2\pi}{\alpha}\ell\right) \times \\ &\quad \sum_{m=0}^{\infty} A_{K-\ell,m} \Gamma\left(m + \frac{2}{\alpha}\ell\right) x^{-m - \frac{2}{\alpha}\ell} \frac{1}{K!} (\lambda_b \pi)^{\frac{1}{2}m\alpha + \ell} \Gamma\left(K - \frac{m\alpha}{2} - \ell + 1, \left(\frac{K}{x}\right)^{\frac{2}{\alpha}} \lambda_b \pi\right). \end{aligned} \quad (18)$$

The coefficients  $A_{i,j}$  are defined recursively as follows

$$A_{i,j} = \frac{1}{j} \sum_{\ell=1}^j (\ell(i+1) - j) \left(\frac{2}{\ell!(2-\ell\alpha)}\right) A_{i,j-\ell} \quad (19)$$

with  $A_{i,0} = 1$  for  $0 < i \leq K$  and  $\Gamma(\cdot, \cdot)$  is the upper incomplete gamma function.

*Proof:* See Appendix B.

Using Lemma 2, the outage probability of the representative link, which we define as the probability that the spectral efficiency is below a threshold, can be evaluated by:

$$\begin{aligned} P(\eta \leq \tau) &\approx P(\eta_{\text{asym}} \leq \tau) \\ &= F\left(\frac{(2^\tau - 1) K^{1-\frac{\alpha}{2}} L^{-\frac{\alpha}{2}} \lambda^{\frac{\alpha}{2}}}{\left[\frac{\alpha}{2\pi^2} \sin\left(\frac{2\pi}{\alpha}\right)\right]^{\frac{\alpha}{2}}}\right). \end{aligned} \quad (20)$$

This expression can be used to analyze the statistical distribution of the spectral efficiency in the network which is highly dependent on the distribution of distances between the representative mobile and the base stations closest to it. While this expression is complicated, it can be evaluated numerically with greater efficiency than running a Monte Carlo simulation for a wide range of parameters. For instance, in the numerical results described in Section IV, the infinite sum in (20) is evaluated to only 10 terms. Moreover, this result also captures the effects of base-station density on the distribution of the spectral efficiency. For instance, when we expand out the expression in (20) using (18), it is straightforward to see that the resulting approximation for CDF of the spectral efficiency on the representative link is a function of  $\frac{\lambda}{\lambda_b}$ , and not the specific values of  $\lambda$  or  $\lambda_b$ , implying scale invariance as expected.

#### IV. NUMERICAL RESULTS

To demonstrate the accuracy of the analysis, we conducted MC simulations of the network topology. The simulations included 30,000 mobiles and 3,000 BSs in a circular area (i.e., the ratio of the densities is  $\lambda_b/\lambda = 0.1$ ). The simulation did not include white noise, and hence the actual values of the densities does not matter. All simulation were performed for a path-loss exponent of  $\alpha = 4$ .

The simulation studied the performance of a mobile in the center of the simulation area over 10,000 network realizations. Fig. 2 demonstrate the convergence of Lemma 1, by plotting the histogram of the normalized SIR:

$$\frac{\text{SIR}}{(KL)^{\alpha/2-1} \mathbf{P}_K} \quad (21)$$

and its predicted asymptotic value according to Lemma 1.

Fig. 2 shows that the normalized SIR can have a variance of several dBs, due to the random locations of BSs and mobiles in each network realization. The prediction of Lemma 1 seems a little bit optimistic compared to the center of the histogram for low number of antennas. But, for higher number of antennas the histogram becomes narrow, as the normalized SIR converges to a constant. In this regime, the prediction of Lemma 1 becomes very accurate.

To further illustrate the accuracy of Lemma 1, Fig. 3 depicts the normalized SIR, (21), as a function of the number of antennas per BS for cluster sizes of  $K = 2, 4$  and 8 BSs. The Error-bars show the range from the mean minus 1 standard deviation to the mean plus 1 standard deviation, according to the MC simulations. The dashed line show the predicted values according to Lemma 1. The figure shows that in all cases, the theoretical asymptotic prediction is well between the error-bars from the simulations. As the number of antennas gets large, the error-bars become very close (indicating the convergence to a deterministic quantity) and the asymptotic theoretical prediction meet the center of the bars.

So far, the figures showed the accuracy of the results mostly with quite high numbers of antennas. Yet, the results can be very useful even with relatively small number of antennas, if we also take into account the distribution of the received power,  $\mathbf{P}_K$ . Fig. 4 compares the CDF of the asymptotic spectral efficiencies predicted by Theorem 1, with the CDF of the asymptotic spectral efficiencies in the MC simulations. As can be seen, even using 50 antenna per BSs give deviation from the theory of

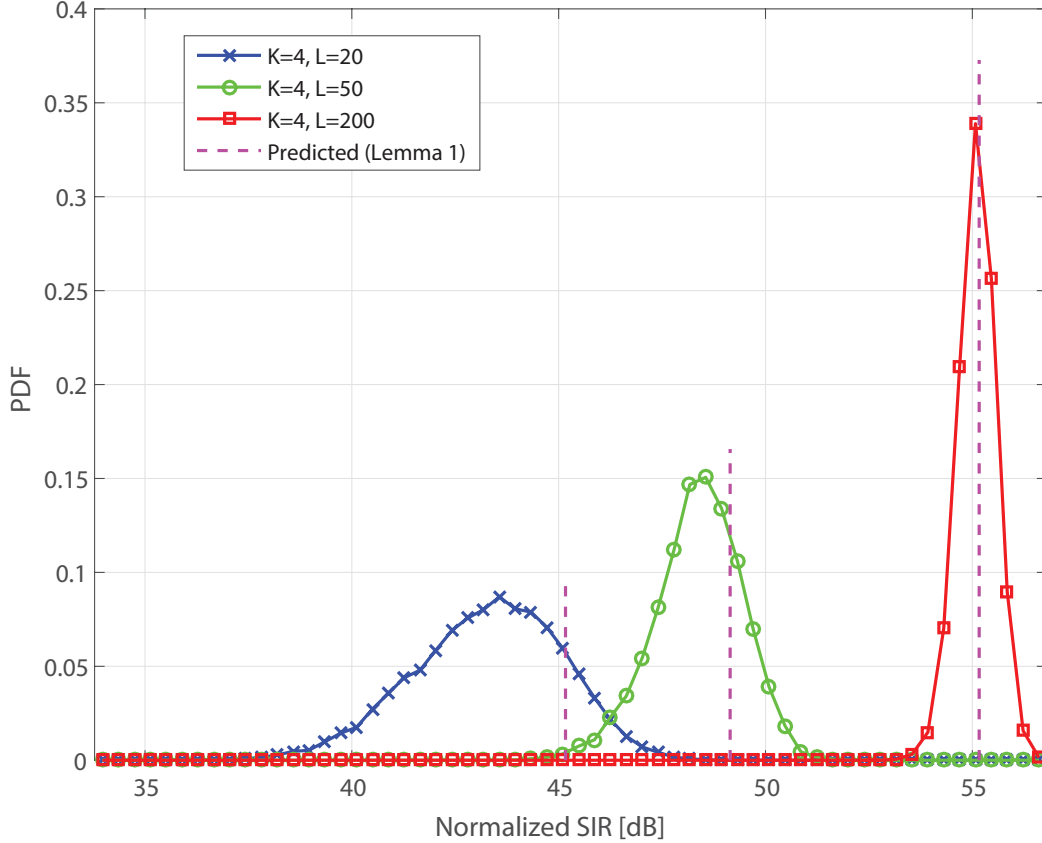


Fig. 2. Histogram of the normalized SIR, (21), using a cluster of the  $K = 4$  nearest BSs and varying number of antennas per BS ( $L$ ). The figure also shows the normalized SIR predicted by Lemma 1.

only fractions of a bps/Hz. (This deviation increases to about 1bps/Hz for 10 antennas per BS). Thus, the result of Theorem 1 gives a reasonable prediction for the achievable performance even in moderately large networks.

As a point of comparison, we have plotted the empirical CDF of the spectral efficiency when a matched filter is used, for the case of 50 antennas per base station and a cluster of 8 base stations. As is evident from the plot, the matched filter performs a lot worse than the MMSE receiver. The reason for this is that our system is interference limited. Moreover, the mobiles transmit with equal power resulting in a large variation in the received powers from the mobiles due to location-dependent path loss. The large discrepancy in received signal powers severely impacts the performance of matched-filter receivers (e.g., see [23]). This significant performance difference between the matched-filter and MMSE receiver indicates that the MMSE receiver is more attractive than the matched-filter not only in scenarios with moderately large numbers of antennas (as described in [14]), but also in distributed antenna systems with a large total number of antennas used, but with a moderately large number of antennas per base station.

## V. SUMMARY AND CONCLUSIONS

In this work, we analyze spatially distributed cellular networks with co-operative base stations which have multiple antennas. Base stations and mobiles are assumed to be distributed according to HPPPs on the plane. We assume a user-centric co-operation approach where the  $K$  base stations closest in Euclidian distance to a test mobile co-operate to detect the transmitted signal from the mobile, in the presence of co-channel interferers. Assuming Gaussian codebooks used by all mobiles, we find an asymptotic expression for the spectral efficiency on a representative link given in (10). This result provides a simple expression for the spectral efficiency that can be achieved in co-operative cellular systems, as a function of tangible system parameters such as density of mobiles, number of co-operating base stations and number of antennas per base station. Moreover, the expression for the CDF of the spectral efficiency (20), provides the probability for a link being in outage, and confirms that in the interference-limited regime, the system is scale invariant in the sense that it is the ratio of mobile to base-station density that matters, rather than their individual values.

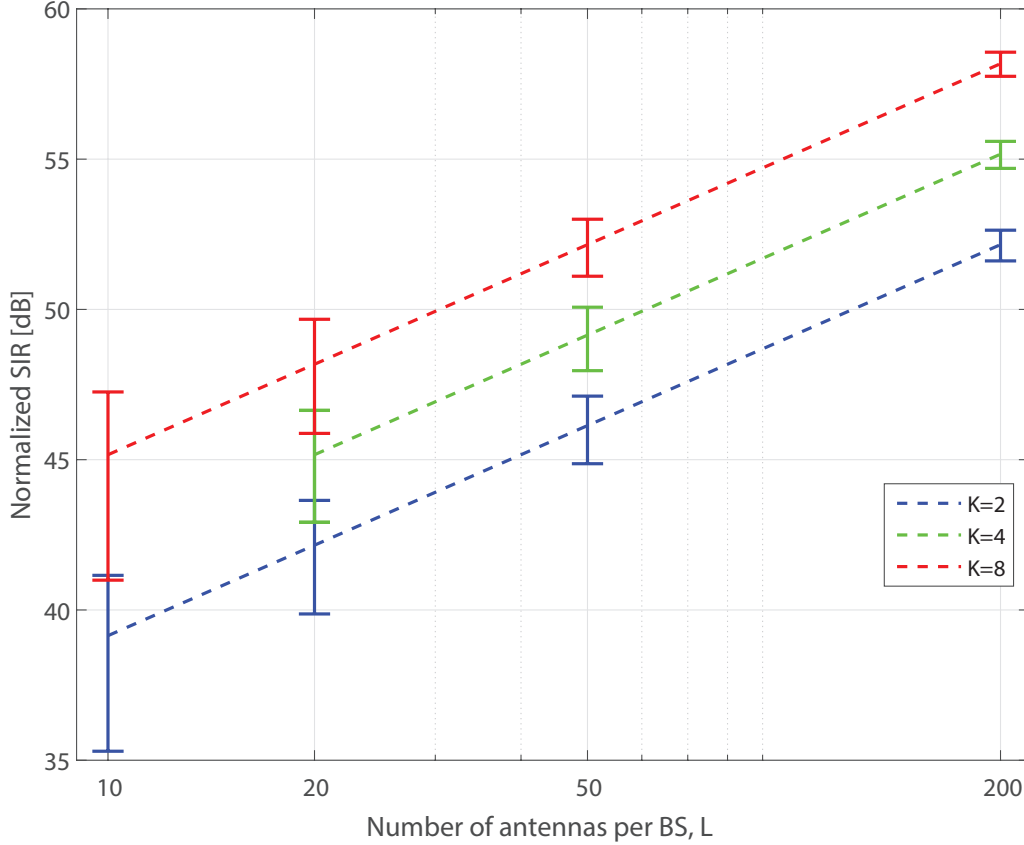


Fig. 3. The normalized SIR, (21), as a function of the number of antennas per BS for various cluster sizes ( $K$ ). The Error-bars show the  $\pm 1$  Std through Monte Carlo (MC) simulations while dashed lines show the predicted values by Lemma 1.

Several insights can be gained from these results. Recall that  $\lambda$  and  $\lambda_b$  are the densities of mobiles and base stations respectively,  $K$  is the number of co-operating base stations,  $L$  is the number of antennas per base station, and  $\alpha$  is the path loss exponent.

In particular, the discussion in Section III-A indicates that an increasing density of mobiles can be supported if the quantity  $\frac{\lambda_b L K^{1-\frac{2}{\alpha}}}{\lambda}$  is kept constant. Since each of the parameters  $\lambda_b$ ,  $K$  and  $L$  is associated with potentially very different implementation complexity and costs, system designers can use this result as a general guideline in making appropriate tradeoffs to support increasing mobile densities by provisioning additional infrastructure.

In future work, the effects of thermal noise, and channel estimation errors should be considered. Since this work indicates the spectral efficiencies that can be achieved if these effects can be kept small enough, further study is required to determine the resources required to achieve sufficiently accurate channel estimates, and the impact of inaccurate channel estimation.

Overall, given the challenges of analytically studying the uplink of spatially distributed co-operative base-station systems which can provide high spectral efficiencies but with significant costs, we expect this result to be useful as it provides simple expressions for the spectral efficiency which capture the effects of the main system parameters.

#### APPENDIX A PROOF OF LEMMA 1

The proof is based on several results in large matrix theory, which were derived for the case that  $n$  and  $L$  grow with a fixed ratio. In the proof of this lemma, we shall consider a system with only the closest  $n$  mobiles to the origin, and then take  $n$  and  $L$  to infinity. The applicability of this approach is guaranteed by the following lemma:

*Lemma 3:* Under the conditions of Theorem 1, if we set  $n = cLK$  with  $c > 2$  then a sufficient condition to guarantee (11) is that the following limit holds in probability:

$$\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} \left| \frac{\text{SIR}[n]}{(KL)^{\alpha/2-1} \mathbf{P}_K} - \left[ \frac{\alpha}{2\pi^2 \lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}} \right| = 0 \quad (22)$$



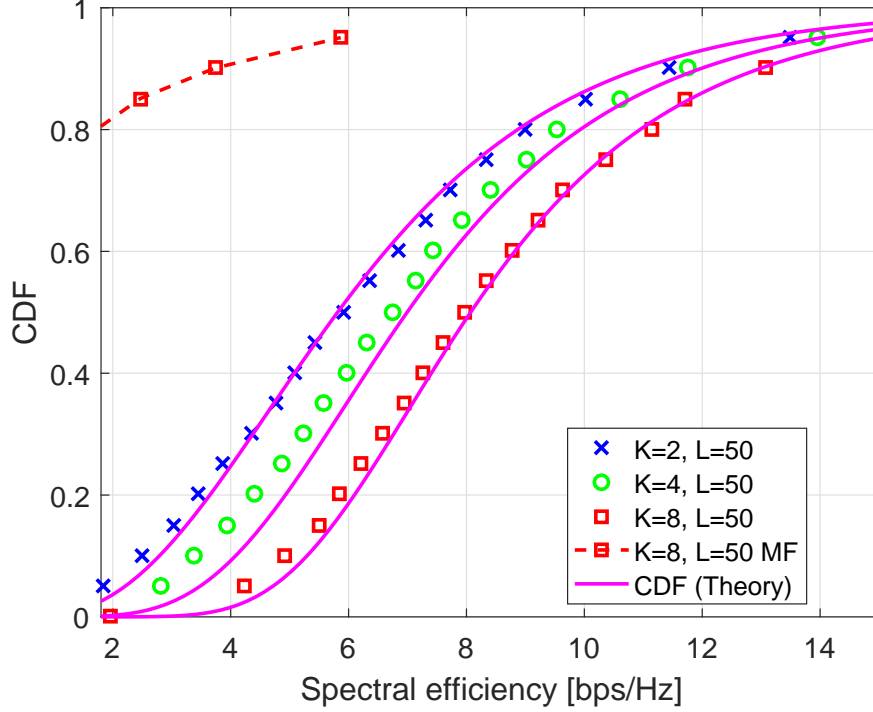


Fig. 4. The CDF of the spectral efficiency in a network with cluster sizes of 2, 4 and 8 BSs, with 50 antennas per BS. Solid lines depict the theoretical result of Theorem 1. Markers depict simulation results.

*Proof:* See Appendix C. ■

Thus, we only need to prove (22). To do this, we shall utilize the following lemma which guarantees that an appropriately scaled version of the matrix in the expression for  $\text{SIR}[n]$  in (5) is invertible for large  $L$  with high probability.

*Lemma 4:* Let  $c_1$  be an integer such that  $1 < c_1 < c$ . Then with probability 1,

$$\lim_{L \rightarrow \infty} \gamma_{\min} \left\{ L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right\} \geq (\pi\lambda)^{\alpha/2} \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2 (c_1 K)^{1-\alpha/2} \triangleq \gamma_{lb} > 0 \quad (23)$$

*Proof:* See Appendix D

Thus, for sufficiently large  $L$ ,  $L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n]$  is invertible with probability 1. Normalizing  $\text{SIR}[n]$  from (5), and assuming the limit on the right-hand side (RHS) below exists, we have the following with probability 1.

$$\lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR}[n] = \lim_{\zeta \rightarrow 0} \lim_{L \rightarrow \infty} \frac{1}{L} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] + \zeta \mathbf{I} \right)^{-1} \mathbf{h}_0 \quad (24)$$

The existence of the limit on the RHS above, and its form are given in the following lemma

*Lemma 5:* Conditioned on  $r_{0,1}, \dots, r_{0,K}$ , as  $L \rightarrow \infty$

$$\frac{1}{LK^{\alpha/2-1}} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] + \zeta \mathbf{I} \right)^{-1} \mathbf{h}_0 \quad (25)$$

converges in probability to a non-random limit, and

$$\lim_{c \rightarrow \infty} \lim_{\zeta \rightarrow 0} \lim_{L \rightarrow \infty} \frac{1}{\mathbf{P}_K K^{\frac{\alpha}{2}-1}} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] + \zeta \mathbf{I} \right)^{-1} \mathbf{h}_0 = \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}} \quad (26)$$

*Proof:* See Appendix E

Combining (26) with (24) yields (22).

## APPENDIX B PROOF OF LEMMA 2

To find the CDF of  $\sum_{i=1}^K r_{0,i}^{-\alpha}$ , we first condition on the event that  $r_{0,(K+1)} = R_{K+1}$ . Since the base stations are distributed according to a HPPP, the  $K$  base stations closest to the representative mobile are distributed with uniform probability in the

interior of the disk of radius  $R_{K+1}$ , centered at the origin. Let  $\tilde{r}_{01}, \tilde{r}_{02}, \dots, \tilde{r}_{0K}$ , be the distances of the  $K$  base stations closest to the representative mobile, in *random* order.  $\tilde{r}_{01}, \tilde{r}_{02}, \dots, \tilde{r}_{0K}$  are i.i.d. in the radius  $R_{K+1}$  disk centered at the origin.

Note that  $R_{K+1}^\alpha \tilde{r}_{0i}^{-\alpha}$  are Pareto distributed random variables with shape parameter  $2/\alpha$ . Since  $\alpha > 2$ , the shape parameter is between zero and unity. For this case, the sum of  $K$  i.i.d. Pareto distributed random variables has the following CDF for  $x \geq K$ ,

$$P\left(\sum_{i=1}^K R_{K+1}^\alpha \tilde{r}_{0i}^{-\alpha} \leq x \mid R_{K+1}\right) = 1 + \frac{1}{\pi} \sum_{\ell=1}^K \binom{K}{\ell} \left(-\Gamma\left(1 - \frac{2}{\alpha}\right)\right)^\ell \sin\left(\frac{2\pi}{\alpha}\ell\right) \sum_{m=0}^{\infty} A_{K-\ell, m} \Gamma\left(m + \frac{2}{\alpha}\ell\right) x^{-m - \frac{2}{\alpha}\ell}, \quad (27)$$

and zero otherwise [24]. From (27), we have

$$P\left(\sum_{i=1}^K \tilde{r}_{0i}^{-\alpha} \leq x \mid R_{K+1}\right) = P\left(\sum_{i=1}^K R_{K+1}^\alpha \tilde{r}_{0i}^{-\alpha} \leq x R_{K+1}^\alpha \mid R_{K+1}\right) = 1 + \frac{1}{\pi} \sum_{\ell=1}^K \binom{K}{\ell} \left(-\Gamma\left(1 - \frac{2}{\alpha}\right)\right)^\ell \sin\left(\frac{2\pi}{\alpha}\ell\right) \sum_{m=0}^{\infty} A_{K-\ell, m} \Gamma\left(m + \frac{2}{\alpha}\ell\right) (x R_{K+1}^\alpha)^{-m - \frac{2}{\alpha}\ell} \quad (28)$$

for  $x R_{K+1}^\alpha \geq K$  and zero otherwise. Since  $R_{K+1}$  is the distance of the  $(K+1)$ st nearest neighbor in a HPPP, its PDF is known and can be found in references such as [25]. To remove the conditioning on  $R_{K+1}$  in (28) with respect to  $R_{K+1}$ , we first take the expectation of the term  $R_{K+1}^{-m\alpha-2\ell} 1_{\{x R_{K+1}^\alpha \geq K\}}$ ,

$$E\left[R_{K+1}^{-m\alpha-2\ell} 1_{\{x R_{K+1}^\alpha \geq K\}}\right] = \int_{\left(\frac{K}{x}\right)^{\frac{1}{\alpha}}}^{\infty} R_{K+1}^{-m\alpha-2\ell} \frac{2(\lambda_b \pi R_{K+1}^2)^{K+1}}{R_{K+1} K!} e^{-\lambda_b \pi R_{K+1}^2} dR_{K+1} \quad (29)$$

Making the substitution  $q = R_{K+1}^2 \lambda_b \pi$ ,  $dq = dR_{K+1} 2R_{K+1} \lambda_b \pi$ , we have

$$\begin{aligned} E\left[R_{K+1}^{-m\alpha-2\ell} 1_{\{x R_{K+1}^\alpha \geq K\}}\right] &= \int_{\left(\frac{K}{x}\right)^{\frac{2}{\alpha}} \lambda_b \pi}^{\infty} (\lambda_b \pi)^{m\alpha/2+\ell} \frac{q^{K-m\alpha/2-\ell}}{K!} e^{-q} dq \\ &= \frac{(\lambda_b \pi)^{m\alpha/2+\ell}}{K!} \Gamma\left(K - \frac{m\alpha}{2} - \ell + 1, \left(\frac{K}{x}\right)^{\frac{2}{\alpha}} \lambda_b \pi\right) \end{aligned} \quad (30)$$

Additionally, applying the CDF of  $R_{K+1}$  which can be found e.g. in [25], we have

$$E\left[1_{\{x R_{K+1}^\alpha \geq K\}}\right] = e^{-\lambda_b \pi \left(\frac{K}{x}\right)^{2/\alpha}} \sum_{\ell=0}^K \frac{\left(\lambda_b \pi \left(\frac{K}{x}\right)^{2/\alpha}\right)^\ell}{\ell!}. \quad (31)$$

Using the previous two expressions to take the expectation of the conditional CDF given in (28) with respect to  $R_{K+1}$ , we have

$$\begin{aligned} F(x) &= P\left(\sum_{i=1}^K \tilde{r}_{0i}^{-\alpha} \leq x\right) = e^{-\lambda_b \pi \left(\frac{K}{x}\right)^{2/\alpha}} \sum_{\ell=0}^K \frac{\left(\lambda_b \pi \left(\frac{K}{x}\right)^{2/\alpha}\right)^\ell}{\ell!} \\ &+ \frac{1}{\pi} \sum_{\ell=1}^K \binom{K}{\ell} \left(-\Gamma\left(1 - \frac{2}{\alpha}\right)\right)^\ell \sin\left(\frac{2\pi}{\alpha}\ell\right) \times \\ &\sum_{m=0}^{\infty} A_{K-\ell, m} \Gamma\left(m + \frac{2}{\alpha}\ell\right) x^{-m - \frac{2}{\alpha}\ell} \frac{1}{K!} (\lambda_b \pi)^{\frac{1}{2}m\alpha+\ell} \Gamma\left(K - \frac{m\alpha}{2} - \ell + 1, \left(\frac{K}{x}\right)^{\frac{2}{\alpha}} \lambda_b \pi\right). \end{aligned} \quad (32)$$

### APPENDIX C PROOF OF LEMMA 3

We need to prove that (22) is a sufficient condition for

$$\lim_{L \rightarrow \infty} \frac{\text{SIR}}{(KL)^{\alpha/2-1} \mathbf{P}_K} - \left[ \frac{\alpha}{2\pi^2 \lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}} = 0. \quad (33)$$

We shall use a sandwiching argument to prove this. First we write an upper bound to the SIR which is based on considering only the first  $n$  mobiles in the network, as compared to SIR which includes an infinite number of mobiles

$$\frac{\text{SIR}}{(KL)^{\alpha/2-1}\mathbf{P}_K} \leq \frac{\text{SIR}[n]}{(KL)^{\alpha/2-1}\mathbf{P}_K} \quad (34)$$

Thus,

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{\text{SIR}}{(KL)^{\alpha/2-1}\mathbf{P}_K} &\leq \lim_{L \rightarrow \infty} \frac{\text{SIR}[n]}{(KL)^{\alpha/2-1}\mathbf{P}_K} = \lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\text{SIR}[n]}{(KL)^{\alpha/2-1}\mathbf{P}_K} \\ &= \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}}, \end{aligned} \quad (35)$$

in probability, where the limit is guaranteed by (22).

Next, consider the MMSE estimator for a network with only  $n$  mobile nodes,  $\mathbf{c}[n]$ . If this estimator is applied to the network with all the mobiles, the resulting SIR, which we define as  $\underline{\text{SIR}}[n]$ , will be a lower bound to the SIR with the full MMSE estimator. Hence,

$$\begin{aligned} \text{SIR} &\geq \underline{\text{SIR}}[n] = \lim_{\ell \rightarrow \infty} \frac{|\mathbf{c}[n]^\dagger \mathbf{h}_0|^2}{\mathbf{c}[n]^\dagger (\mathbf{H}[\ell] \mathbf{H}^\dagger[\ell]) \mathbf{c}[n]} \\ &= \lim_{\ell \rightarrow \infty} \frac{|\mathbf{c}[n]^\dagger \mathbf{h}_0|^2}{\mathbf{c}[n]^\dagger (\mathbf{H}[n] \mathbf{H}^\dagger[n] + \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell]) \mathbf{c}[n]} \\ &= \lim_{\ell \rightarrow \infty} \frac{|\mathbf{c}[n]^\dagger \mathbf{h}_0|^2}{\mathbf{c}[n]^\dagger \mathbf{H}[n] \mathbf{H}^\dagger[n] \mathbf{c}[n] \left(1 + \frac{\mathbf{c}^\dagger[n] \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n]}{\mathbf{c}[n]^\dagger \mathbf{H}[n] \mathbf{H}^\dagger[n] \mathbf{c}[n]}\right)} \\ &= \frac{\text{SIR}[n]}{1 + \lim_{\ell \rightarrow \infty} \frac{\mathbf{c}^\dagger[n] \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n]}{\text{SIR}[n]}} \end{aligned} \quad (36)$$

where  $\mathbf{\Delta}_n[\ell] = [\mathbf{h}_{n+1} | \dots | \mathbf{h}_\ell]$ . Next consider the following

$$\lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR} = \lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR} \geq \frac{\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR}[n]}{1 + \frac{\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \lim_{\ell \rightarrow \infty} \mathbf{c}^\dagger[n] \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n]}{\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR}[n]}} \quad (38)$$

Since  $\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR}[n] = K^{\alpha/2-1} \sum_{k=1}^K r_{0,k}^{-\alpha} \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}}$ , we have

$$\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \text{SIR} \geq \frac{K^{\alpha/2-1} \sum_{k=1}^K r_{0,k}^{-\alpha} \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}}}{1 + \frac{\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} L^{-\alpha/2} \lim_{\ell \rightarrow \infty} \mathbf{c}^\dagger[n] \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n]}{K^{\alpha/2-1} \sum_{k=1}^K r_{0,k}^{-\alpha} \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}}}} \quad (39)$$

The denominator of the RHS of the previous expression converges to unity as a consequence of the following lemma.

*Lemma 6:*

$$\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\ell \rightarrow \infty} L^{-\alpha/2} \mathbf{c}^\dagger[n] \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n] = 0 \quad (40)$$

in probability.

*Proof: Please see Appendix F*

Hence the following holds in probability,

$$\lim_{L \rightarrow \infty} \frac{\text{SIR}}{(KL)^{\alpha/2-1}\mathbf{P}_K} \geq \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}}. \quad (41)$$

From (35) we observe that the upper and lower bounds converge to the same quantity in probability. Hence we conclude that

$$\lim_{L \rightarrow \infty} \frac{\text{SIR}}{(KL)^{\alpha/2-1}\mathbf{P}_K} = \left[ \frac{\alpha}{2\pi^2\lambda} \sin\left(\frac{2\pi}{\alpha}\right) \right]^{\frac{\alpha}{2}} \quad (42)$$

in probability.

APPENDIX D  
PROOF OF LEMMA 4

First let's define  $M = c_1 KL$ , where  $c_1$  is a constant, with  $1 < c_1 < c$ . Since  $n = cKL > M$ , by the Weyl inequality (see e.g. [26]),

$$\gamma_{\min} \left( M^{\alpha/2-1} \mathbf{H}[M] \mathbf{H}^\dagger[M] \right) \leq \gamma_{\min} \left( M^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right) \quad (43)$$

whereby,

$$\begin{aligned} \mathbf{H}[M] \mathbf{H}^\dagger[M] &= \sum_{i=1}^M \mathbf{h}_i \mathbf{h}_i^\dagger \\ &= \sum_{i=1}^M (r_{M+1} + R_{K+1})^{-\alpha} \mathbf{g}_i \mathbf{g}_i^\dagger + \left[ \sum_{i=1}^M (\mathbf{h}_i \mathbf{h}_i^\dagger - (r_{M+1} + R_{K+1})^{-\alpha} \mathbf{g}_i \mathbf{g}_i^\dagger) \right] \end{aligned} \quad (44)$$

where  $\mathbf{g}_i = (g_{i1}, g_{i2}, \dots, g_{iN})^T$ . Recall that

$$\mathbf{h}_i = \left( g_{i,1} \cdot r_{i,1}^{-\alpha/2}, \dots, g_{i,L} \cdot r_{i,1}^{-\alpha/2}, g_{i,L+1} \cdot r_{i,2}^{-\alpha/2}, \dots, g_{i,2L} \cdot r_{i,2}^{-\alpha/2}, \dots, g_{i,LK} \cdot r_{i,K}^{-\alpha/2} \right)^T. \quad (45)$$

Since  $r_{i,j} < r_{M+1} + R_{K+1}$  for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, K$  the matrix in the brackets on the RHS of (44) is non-negative definite with probability 1. Hence, by the Weyl inequality (see e.g. [26])

$$\gamma_{\min} \left\{ \sum_{i=1}^M (r_{M+1} + R_{K+1})^{-\alpha} \mathbf{g}_i \mathbf{g}_i^\dagger \right\} \leq \gamma_{\min} \{ \mathbf{H}[M] \mathbf{H}^\dagger[M] \} \quad (46)$$

From [27], the following is known to hold with probability 1:

$$\lim_{M \rightarrow \infty} \gamma_{\min} \left\{ \sum_{i=1}^M \frac{1}{M} \mathbf{g}_i \mathbf{g}_i^\dagger \right\} = \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2. \quad (47)$$

Additionally, since  $r_{M+1}$  is a scaled  $\chi^2$  random variable with  $2M + 2$  degrees of freedom (see e.g. [25] Equation 2.4.4), the following holds in the mean-square sense

$$\lim_{M \rightarrow \infty} \frac{\pi \lambda r_{M+1}^2}{M + 1} = 1. \quad (48)$$

Therefore the following holds in the mean-square sense,

$$\lim_{M \rightarrow \infty} (r_{M+1} + R_{K+1})^{-\alpha} \left( \frac{M}{\pi \lambda} \right)^{\alpha/2} = 1 \quad (49)$$

Since mean-square convergence and convergence with probability 1 both imply convergence in probability, combining (47) and (49), we have

$$\lim_{M \rightarrow \infty} M^{\alpha/2-1} \gamma_{\min} \left\{ (r_{M+1} + R_{K+1})^{-\alpha} \sum_{i=1}^M \mathbf{g}_i \mathbf{g}_i^\dagger \right\} = (\pi \lambda)^{\alpha/2} \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2 \quad (50)$$

in probability. Combining this expression with (46) and (43) yields the following in probability

$$\begin{aligned} (\pi \lambda)^{\alpha/2} \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2 &\leq \lim_{L \rightarrow \infty} \gamma_{\min} \left\{ (c_1 KL)^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right\} \\ (\pi \lambda)^{\alpha/2} \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2 (c_1 K)^{1-\alpha/2} &\leq \lim_{L \rightarrow \infty} \gamma_{\min} \left\{ L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right\} \end{aligned} \quad (51)$$

APPENDIX E  
PROOF OF LEMMA 5

To prove this lemma, we use a prior result from [28]. In order to use this result, we define  $q_{i,j}$  as i.i.d. random variables taking values of  $\pm 1$  with equal probability, and  $\bar{g}_{i,j} = g_{i,j}/q_{i,j}$ . Note that  $\bar{g}_{i,j}$  are still i.i.d  $\mathcal{CN}(0, 1)$  random variables. We thus have

$$\begin{aligned} \mathbf{h}_i &= \left( \bar{g}_{i,1} \cdot q_{i,1} \cdot r_{i,1}^{-\alpha/2}, \dots, \bar{g}_{i,L} \cdot q_{i,1} \cdot r_{i,1}^{-\alpha/2}, \bar{g}_{i,L+1} \cdot q_{i,2} \cdot r_{i,2}^{-\alpha/2}, \dots, \right. \\ &\quad \left. \bar{g}_{i,2L} \cdot q_{i,2} \cdot r_{i,2}^{-\alpha/2}, \dots, \bar{g}_{i,LK} \cdot q_{i,K} \cdot r_{i,K}^{-\alpha/2} \right)^T. \end{aligned} \quad (52)$$

Next we define the empirical distribution function (e.d.f.) of  $N^{\alpha/2}|q_{i,j}r_{i,j}^{-\alpha/2}|^2 = N^{\alpha/2}r_{i,j}^{-\alpha}$  as the function  $H_N(\tau_1, \dots, \tau_K)$  below

$$H_N(\tau_1, \dots, \tau_K) = \frac{1}{n} \sum_{i=1}^n 1_{\{N^{\alpha/2}r_{i,1}^{-\alpha} \leq \tau_1, \dots, N^{\alpha/2}r_{i,K}^{-\alpha} \leq \tau_K\}}. \quad (53)$$

Hence,  $H_N(\tau_1, \dots, \tau_K)$  measures the proportion of the  $n$  mobiles for which the quantities  $N^{\alpha/2}r_{i,1}^{-\alpha} \leq \tau_1, \dots, N^{\alpha/2}r_{i,K}^{-\alpha}$  are less than or equal to  $\tau_1, \dots, \tau_K$ , respectively. From Corollary 1.2 of [28] if the following two conditions are true,

- 1)  $H_N(\tau_1, \dots, \tau_K)$  converges with probability 1 to a  $K$  dimensional, non-random function  $H(\tau_1, \tau_2, \dots, \tau_K)$ ,
- 2) For all  $\ell \neq \ell'$ , and all positive  $\nu_1, \dots, \nu_K$ , the following holds with probability 1,

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} \sum_{i=1}^n \frac{q_{i,\ell} q_{i,\ell'} N^{\alpha/2} r_{i,\ell}^{-\alpha/2} r_{i,\ell'}^{-\alpha/2}}{1 + \sum_{j=1}^K \nu_j N^{\alpha/2} r_{i,j}^{-\alpha}} = 0, \quad (54)$$

then, with probability 1,

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{LK^{\alpha/2-1}} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] + \zeta \mathbf{I} \right)^{-1} \mathbf{h}_0 &= \lim_{L \rightarrow \infty} \frac{1}{L} \mathbf{h}_0^\dagger \left( N^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] + \zeta \mathbf{I} \right)^{-1} \mathbf{h}_0 \\ &= \sum_{i=1}^L r_{0,i}^{-\alpha} a_i, \end{aligned} \quad (55)$$

where  $a_i$  for  $i = 1, 2, \dots, K$ , is given uniquely by the set of  $a_i$ s satisfying the following equations for  $i = 1, 2, \dots, K$ .

$$a_i = \frac{1}{\frac{n}{L} E \left[ \frac{\tau_i}{1 + \sum_{j=1}^K a_j \tau_j} \right] + \zeta}. \quad (56)$$

The  $\tau_i$  terms are random variables with joint CDF  $H(\tau_1, \tau_2, \dots, \tau_K)$ .

The convergence of the e.d.f. of  $N^{\alpha/2}r_{i,j}^{-\alpha}$  required for condition 1) above is proved in the first part of following lemma, and the second part used in evaluating (56).

*Lemma 7:* As  $N \rightarrow \infty$ ,  $H_N(\tau_1, \dots, \tau_K)$ , converges with probability 1 to a joint CDF  $H(\tau_1, \tau_2, \dots, \tau_K)$ . Moreover, if  $\tau_1, \tau_2, \dots, \tau_K$  are random variables with joint CDF  $H(\tau_1, \dots, \tau_K)$ , the CDF of  $\tau = \frac{\tau_1 + \tau_2 + \dots + \tau_K}{K}$ , equals  $G(\tau)$  given below

$$G(\tau) = 1 - \frac{\pi \lambda}{c} (\tau)^{-\frac{2}{\alpha}} 1_{\left\{ \left( \frac{\pi \lambda p}{c} \right)^{\alpha/2} < \tau \right\}}. \quad (57)$$

*Proof:* Please see Appendix G.

Condition 2) can be proved as follows.

$$\begin{aligned} \left| \frac{1}{n-1} \sum_{i=q}^n \frac{q_{i,\ell} q_{i,\ell'} N^{\alpha/2} r_{i,\ell}^{-\alpha/2} r_{i,\ell'}^{-\alpha/2}}{1 + \sum_{j=1}^K \nu_j N^{\alpha/2} r_{i,j}^{-\alpha}} \right| &\leq \left| \frac{1}{n-1} \sum_{i=q}^n \frac{q_{i,\ell} q_{i,\ell'} (\max_{\ell} \{r_{i,\ell}\})^{-\alpha}}{\sum_{j=1}^K \nu_j r_{i,j}^{-\alpha}} \right| \leq \\ &\left| \frac{1}{n-1} \sum_{i=q}^n \frac{q_{i,\ell} q_{i,\ell'} (\max_{\ell} \{r_{i,\ell}\})^{-\alpha}}{\min_k \{\nu_k\} \sum_{j=1}^K r_{i,j}^{-\alpha}} \right| \leq \frac{1}{\min_k \{\nu_k\}} \left| \frac{1}{n-1} \sum_{i=q}^n q_{i,\ell} q_{i,\ell'} \right| \end{aligned} \quad (58)$$

The last inequality follows from the fact that the summation in the denominator of the term on the left-hand side (LHS) is over all  $j$ , and hence must include the term  $\max_{\ell} \{r_{i,\ell}\}$ . As  $n, N \rightarrow \infty$ , for  $\ell \neq \ell'$ , the RHS above converges to zero with probability 1 by the strong law of large numbers.

To find  $a_i$ , observe that by symmetry, and the fact that (56) has a unique solution,  $a_1 = a_2 = \dots = a_K = a$ . Thus, we can write (56) as

$$\begin{aligned} a &= \frac{1}{\frac{n}{L} E \left[ \frac{\tau_i}{1 + a \sum_{j=1}^K \tau_j} \right] + \zeta} = \frac{1}{\frac{n}{L} E \left[ \frac{\frac{1}{K} \sum_{k=1}^K \tau_k}{1 + \frac{a}{K} \sum_{j=1}^K \tau_j} \right] + \zeta} \\ &= \frac{1}{\frac{n}{L} E \left[ \frac{\tau}{1 + a K \tau} \right] + \zeta} \end{aligned} \quad (59)$$

where  $\tau$  is as defined in Lemma 7. Recalling that  $a_1 = a_2 = \dots = a$ , we have from (55) that

$$\lim_{L \rightarrow \infty} \frac{1}{L} \mathbf{h}_0^\dagger \left( N^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] + \zeta \mathbf{I} \right)^{-1} \mathbf{h}_0 = a \sum_{i=1}^L r_{0,i}^{-\alpha} \quad (60)$$

Recalling that  $N = LK$  and  $c = n/N$ , if we rewrite (59) as

$$(aK)cE \left[ \frac{\tau}{1 + (aK)\tau} \right] + a\zeta = 1, \quad (61)$$

we can use a prior result from [29] where the uplink of non-co-operative base stations was analyzed (i.e. the case when  $K = 1$ ). From [29], it is known that the following holds

$$\lim_{c \rightarrow \infty} \lim_{\zeta \rightarrow 0} aK = \left[ \frac{\alpha}{2\pi^2 \lambda} \sin \left( \frac{2\pi}{\alpha} \right) \right]^{\frac{\alpha}{2}}. \quad (62)$$

Taking the limits as  $\zeta \rightarrow 0$  followed by  $c \rightarrow \infty$  of (60), substituting the previous expression, and the definition of  $\mathbf{P}_K$ , completes the proof.

## APPENDIX F PROOF OF LEMMA 6

First, we define  $\epsilon[n, \ell] = \mathbf{c}^\dagger[n] \mathbf{\Delta}_n[\ell] \mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n]$  and note that:

$$\begin{aligned} \epsilon[n, \ell] &= \|\mathbf{\Delta}_n^\dagger[\ell] \mathbf{c}[n]\|^2 \\ &= \sum_{m=n+1}^{\ell} |\mathbf{h}_m^\dagger \mathbf{c}[n]|^2 \\ \lim_{\ell \rightarrow \infty} \epsilon[n, \ell] &= \epsilon[n] \triangleq \sum_{m=n+1}^{\infty} |\mathbf{h}_m^\dagger \mathbf{c}[n]|^2 \end{aligned} \quad (63)$$

Also note that

$$\mathbf{h}_m^\dagger \mathbf{c}[n] = \sum_{k=1}^K |\mathbf{h}_{m,k}^\dagger \mathbf{c}_k[n]|^2 \quad (64)$$

where  $\mathbf{h}_{m,k}^\dagger$  and  $\mathbf{c}_k[n]$  are the sub-vectors associated with the  $k$ -th BS.

Given  $\mathbf{c}_k[n]$ , the product  $\mathbf{h}_{m,k}^\dagger \mathbf{c}_k[n]$  is a complex Gaussian random variable with zero mean and a variance of  $r_{m,k}^{-\alpha} \|\mathbf{c}_k[n]\|^2$  which is statistically independent of any other product in the summations of (63) and (64). Thus, we can define

$$q_{m,k} = \frac{\mathbf{h}_{m,k}^\dagger \mathbf{c}_k[n]}{r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\|} \quad (65)$$

and note that given  $\mathbf{c}_k[n]$ , these are iid random variables with complex Gaussian distribution of zero mean and unit variance.

Now,

$$\epsilon[n] = \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 \quad (66)$$

Next, since  $r_{m,k} > r_m - R_k$  whenever  $r_m > R_k$ , we can bound  $\epsilon[n]$  as follows.

$$\begin{aligned} \epsilon[n] &\leq \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| (r_m - R_k)^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 1_{\{r_m > R_K\}} \\ &\quad + \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 1_{\{r_m \leq R_K\}} \\ &= \|\mathbf{c}[n]\|^2 \sum_{m=n+1}^{\infty} (r_m - R_K)^{-\alpha} |q_m|^2 1_{\{r_m > R_K\}} + \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 1_{\{r_m \leq R_K\}} \end{aligned} \quad (67)$$

where

$$q_m = \frac{\sum_{k=1}^K \|\mathbf{c}_k[n]\| \cdot q_{m,k}}{\|\mathbf{c}[n]\|} \quad (68)$$

is also complex Gaussian with zero mean and unit norm.

Let's now bound the probability that a scaled version of the second term on the RHS of the above equation is greater than zero.

$$\begin{aligned} \Pr \left\{ n^{\alpha/2-1} \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 1_{\{r_m \leq R_K\}} > 0 \right\} &\leq \Pr(|i : r_i \leq R_K \text{ and } i > n| > 0) \\ &= \Pr(r_{n+1} \leq R_K) \\ \lim_{n \rightarrow \infty} \Pr \left\{ n^{\alpha/2-1} \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 1_{\{r_m \leq R_K\}} > 0 \right\} &= 0, \end{aligned}$$

implying the following in probability

$$\lim_{n \rightarrow \infty} n^{\alpha/2-1} \sum_{m=n+1}^{\infty} \sum_{k=1}^K \left| r_{m,k}^{-\alpha/2} \|\mathbf{c}_k[n]\| \cdot q_{m,k} \right|^2 1_{\{r_m \leq R_K\}} = 0. \quad (69)$$

In [30] it is shown that

$$\lim_{n \rightarrow \infty} n^{\alpha/2-1} \sum_{m=n+1}^{\infty} (r_m - R_K)^{-\alpha} |q_m|^2 1_{\{r_m > R_K\}} = \frac{2(\pi\lambda)^{\alpha/2}}{(\alpha - 2)} \quad (70)$$

Combining (67), (69), (70), and the previous expression implies the following in probability

$$\lim_{n \rightarrow \infty} \lim_{\ell \rightarrow \infty} n^{\alpha/2-1} \frac{\mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n]}{\|\mathbf{c}[n]\|^2} \leq \frac{2(\pi\lambda)^{\alpha/2}}{(\alpha - 2)}. \quad (71)$$

Next, consider the following

$$\begin{aligned} L^{-\alpha/2} \mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n] &= L^{-\alpha/2} n^{1-\alpha/2} \|\mathbf{c}[n]\|^2 \left( n^{\alpha/2-1} \frac{\mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n]}{\|\mathbf{c}[n]\|^2} \right) \\ &= L^{1-\alpha} (cK)^{1-\alpha/2} \|\mathbf{c}[n]\|^2 \left( n^{\alpha/2-1} \frac{\mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n]}{\|\mathbf{c}[n]\|^2} \right) \\ \lim_{L \rightarrow \infty} \lim_{\ell \rightarrow \infty} L^{-\alpha/2} \mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n] &\leq \lim_{L \rightarrow \infty} L^{1-\alpha} (cK)^{1-\alpha/2} \|\mathbf{c}[n]\|^2 \frac{2(\pi\lambda)^{\alpha/2}}{(\alpha - 2)}, \end{aligned} \quad (72)$$

where the last limit is from substituting (71). Recall that  $\|\mathbf{c}[n]\|^2 = \mathbf{h}_0^\dagger (\mathbf{H}[n] \mathbf{H}^\dagger[n])^{-2} \mathbf{h}_0$ . Hence

$$\begin{aligned} \lim_{L \rightarrow \infty} \lim_{\ell \rightarrow \infty} L^{-\alpha/2} \mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n] &\leq \lim_{L \rightarrow \infty} L^{1-\alpha} (cK)^{1-\alpha/2} \mathbf{h}_0^\dagger (\mathbf{H}[n] \mathbf{H}^\dagger[n])^{-2} \mathbf{h}_0 \frac{2(\pi\lambda)^{\alpha/2}}{(\alpha - 2)}, \\ &= \lim_{L \rightarrow \infty} (cK)^{1-\alpha/2} \frac{1}{L} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-2} \mathbf{h}_0 \frac{2(\pi\lambda)^{\alpha/2}}{(\alpha - 2)}. \end{aligned} \quad (73)$$

Next, consider the following

$$\frac{1}{L} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-2} \mathbf{h}_0 \leq \frac{1}{(\gamma_{\min} \{L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n]\})^2} \frac{1}{L} \mathbf{h}_0^\dagger \mathbf{h}_0 \quad (74)$$

Applying Lemma 4 to (74), and taking the limit as  $L \rightarrow \infty$  yields

$$\lim_{L \rightarrow \infty} \frac{1}{L} \mathbf{h}_0^\dagger \left( L^{\alpha/2-1} \mathbf{H}[n] \mathbf{H}^\dagger[n] \right)^{-2} \mathbf{h}_0 \leq \frac{\sum_{k=1}^K r_{0,k}^{-\alpha}}{\left( (\pi\lambda)^{\alpha/2} \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2 (c_1 K)^{1-\alpha/2} \right)^2} \quad (75)$$

Substituting the previous expression into (73)

$$\lim_{L \rightarrow \infty} \lim_{\ell \rightarrow \infty} L^{-\alpha/2} \mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n] \leq (cK)^{1-\alpha/2} \frac{2(\pi\lambda)^{\alpha/2} \sum_{k=1}^K r_{0,k}^{-\alpha}}{\left( (\pi\lambda)^{\alpha/2} \left( 1 - \sqrt{\frac{1}{c_1}} \right)^2 (c_1 K)^{1-\alpha/2} \right)^2 (\alpha - 2)},$$

Since  $\alpha > 2$ , and  $c_1 > 1$  is a constant, we have

$$\lim_{c \rightarrow \infty} \lim_{L \rightarrow \infty} \lim_{\ell \rightarrow \infty} L^{-\alpha/2} \mathbf{c}^\dagger[n] \Delta_n[\ell] \Delta_n[\ell]^\dagger \mathbf{c}[n] = 0 \quad (76)$$

APPENDIX G  
PROOF OF LEMMA 7

Let  $\tilde{r}_i$ ,  $i = 1, 2, \dots, n$  be the distances from the origin of the  $n$  closest mobiles to the origin, in random order, and  $\tilde{r}_{i,j}$  be the distance of the corresponding mobile from the  $j$ -th base station, for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, K$ . Because the mobiles are distributed according to a HPPP, conditioned on the distance of the  $(n+1)$ th mobile from the origin,  $r_{n+1}$ , the  $n$  mobiles closest to the origin are distributed with uniform probability in a disk of radius  $r_{n+1}$  centered at the origin. Since the  $K$  base stations closest to the mobile at the origin are at most a distance  $R_K$  from the origin, conditioned on the distance of the  $(n+1)$ th mobile from the origin,  $r_{n+1}$ , we have

$$\begin{aligned} \Pr(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau) &\leq \Pr\left(N^{\alpha/2} (\tilde{r}_i + R_K)^{-\alpha} \leq \tau\right) = \Pr(\tilde{r}_i \geq \tau^{-1/\alpha} N^{1/2} - R_K) \\ &= 1 - \frac{\left(\tau^{-1/\alpha} - \frac{R_K}{\sqrt{N}}\right)^2 N}{r_{n+1}^2} 1_{\left\{\left(\tau^{-1/\alpha} - \frac{R_K}{\sqrt{N}}\right)^{-\alpha} > N^{\alpha/2} r_{n+1}^{-\alpha}\right\}} \end{aligned} \quad (77)$$

$$= 1 - \frac{\left(\tau^{-1/\alpha} - \frac{R_K}{\sqrt{N}}\right)^2 n}{c r_{n+1}^2} 1_{\left\{\left(\tau^{-1/\alpha} - \frac{R_K}{\sqrt{N}}\right)^{-\alpha} > \left(\frac{n}{c r_{n+1}^2}\right)^{\frac{\alpha}{2}}\right\}} \quad (78)$$

Since  $2\pi\lambda r_{n+1}^2$  is a  $\chi^2$  random variable with  $2n+2$  degrees of freedom (see e.g. [25] Equation 2.4.4), it can be written as the sum of the squares of  $2n+2$  i.i.d. standard normal random variables, whose sample mean equals  $\frac{2\pi\lambda r_{n+1}^2}{2n+2}$ . Hence, by the strong law of large numbers, we have the following with probability 1

$$\lim_{n \rightarrow \infty} \frac{r_{n+1}^2}{n} = \frac{1}{\pi\lambda}. \quad (79)$$

Substituting (79) into (78), we get the following with probability 1

$$\lim_{N \rightarrow \infty} \Pr(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau) \leq 1 - \frac{\tau^{-2/\alpha} \pi\lambda}{c} 1_{\left\{\left(\frac{\pi\lambda}{c}\right)^{\alpha/2} < \tau\right\}} \quad (80)$$

For a lower bound, we can write

$$\begin{aligned} \Pr(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau) &\geq \Pr\left(N^{\alpha/2} (\tilde{r}_i - R_K)^{-\alpha} \leq \tau \mid \tilde{r}_i > R_K\right) \Pr(\tilde{r}_i > R_K) \\ &\quad + \Pr\left(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau \mid \tilde{r}_i \leq R_K\right) \Pr(\tilde{r}_i \leq R_K) \end{aligned} \quad (81)$$

Since as  $n \rightarrow \infty$ ,  $\Pr(\tilde{r}_i \leq R_K) \rightarrow 0$  with probability 1, the second term on the RHS of the expression above goes to zero with probability 1 as well. Hence we only need to consider the first term on the RHS in the limit.

$$\Pr\left(N^{\alpha/2} (\tilde{r}_i - R_K)^{-\alpha} \leq \tau \mid \tilde{r}_i > R_K\right) = 1 - \frac{\left(\tau^{-1/\alpha} + \frac{R_K}{\sqrt{N}}\right)^2 n}{c r_{n+1}^2} 1_{\left\{\left(\tau^{-1/\alpha} + \frac{R_K}{\sqrt{N}}\right)^{-\alpha} > \left(\frac{n}{c r_{n+1}^2}\right)^{\frac{\alpha}{2}}\right\}} \quad (82)$$

Hence, we get the following with probability 1,

$$\lim_{N \rightarrow \infty} \Pr\left(N^{\alpha/2} (\tilde{r}_i - R_K)^{-\alpha} \leq \tau \mid \tilde{r}_i > R_K\right) \geq 1 - \frac{\tau^{-2/\alpha} \pi\lambda}{c} 1_{\left\{\left(\frac{\pi\lambda}{c}\right)^{\alpha/2} < \tau\right\}} \quad (83)$$

Since as  $n, N \rightarrow \infty$ ,  $\Pr(\tilde{r}_i > R_K) \rightarrow 1$ , taking the limit of (81) and substituting (83) yields the following with probability 1.

$$\lim_{N \rightarrow \infty} \Pr(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau) \geq 1 - \frac{\tau^{-2/\alpha} \pi\lambda}{c} 1_{\left\{\left(\frac{\pi\lambda}{c}\right)^{\alpha/2} < \tau\right\}} \quad (84)$$

which together with (80) implies the following with probability 1

$$\lim_{N \rightarrow \infty} \Pr(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau) = 1 - \frac{\tau^{-2/\alpha} \pi\lambda}{c} 1_{\left\{\left(\frac{\pi\lambda}{c}\right)^{\alpha/2} < \tau\right\}} \triangleq F_r(\tau). \quad (85)$$

By Proposition 5.25 of [31], we have the following with probability 1, for some joint CDF  $F_{\mathbf{r}}(\tau_1, \dots, \tau_K)$ ,

$$\lim_{N \rightarrow \infty} \Pr(N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau_1, \dots, N^{\alpha/2} \tilde{r}_{i,j}^{-\alpha} \leq \tau_K) = F_{\mathbf{r}}(\tau_1, \dots, \tau_K). \quad (86)$$

From (85), as  $N \rightarrow \infty$  we have the following

$$\Pr\left(H_N(\tau_1, \dots, \tau_K) = F_{\mathbf{r}}(\tau_1, \dots, \tau_K), \left|\Pr(N^{\alpha/2} \tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1)\right| \geq \frac{\epsilon}{2}\right) \rightarrow 0. \quad (87)$$



Next we denote the e.d.f. of  $N^{\alpha/2}\tilde{r}_{i,j}^{-\alpha}$  by  $H_N(\tau)$ . We can then write

$$\begin{aligned} & \Pr \left\{ |H_N(\tau_1, \dots, \tau_K) - F_{\mathbf{r}}(\tau_1, \dots, \tau_K)| > \epsilon, \left| \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1) \right| < \frac{\epsilon}{2} \right\} \\ & \leq \Pr \left\{ |H_N(\tau_1) - F_r(\tau_1)| > \epsilon \left| \left| \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1) \right| < \frac{\epsilon}{2} \right| \right\} \\ & \leq \Pr \left\{ |H_N(\tau_1) - \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha}(\tau_1))| > \frac{\epsilon}{2} \left| \left| \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1) \right| < \frac{\epsilon}{2} \right| \right\} \end{aligned} \quad (88)$$

By the Dvoretzky-Kiefer-Wolfowitz inequality, for any  $\epsilon > 0$  and all  $N$ , we have

$$\Pr \left\{ |H_N(\tau_1) - \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha}(\tau_1))| > \frac{\epsilon}{2} \left| \left| \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1) \right| < \frac{\epsilon}{2} \right| \right\} \leq 2e^{-\frac{N\epsilon^2}{2}} \quad (89)$$

Substituting into (88) yields

$$\Pr \left\{ |H_N(\tau_1, \dots, \tau_K) - F_{\mathbf{r}}(\tau_1, \dots, \tau_K)| > \epsilon, \left| \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1) \right| < \frac{\epsilon}{2} \right\} \leq 2e^{-\frac{N\epsilon^2}{2}}$$

which implies that

$$\sum_{N=0}^{\infty} \Pr \left\{ |H_N(\tau_1, \dots, \tau_K) - F_{\mathbf{r}}(\tau_1, \dots, \tau_K)| > \epsilon, \left| \Pr(N^{\alpha/2}\tilde{r}_{1,j}^{-\alpha} \leq \tau_1) - F_r(\tau_1) \right| < \frac{\epsilon}{2} \right\} < \infty.$$

By Proposition 5.7 of [31], as  $N \rightarrow \infty$ ,

$$\Pr \left\{ |H_N(\tau_1, \dots, \tau_K) - F_{\mathbf{r}}(\tau_1, \dots, \tau_K)| > \frac{\epsilon}{2} \right\} \rightarrow 1. \quad (90)$$

Adding the previous expression and (87), leads to the following with probability 1.

$$\lim_{N \rightarrow \infty} H_N(\tau_1, \dots, \tau_K) = F_{\mathbf{r}}(\tau_1, \dots, \tau_K) = H(\tau_1, \dots, \tau_K). \quad (91)$$

Thus, the first part of the lemma is proved.

For the second part, consider the following:

$$\begin{aligned} & \Pr \left\{ N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha} + \dots + N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha} \leq K\tau | r_i > R_K \right\} = \Pr \left\{ r_{i,1}^{-\alpha} + \dots + r_{i,K}^{-\alpha} \leq K\tau N^{-\frac{\alpha}{2}} | r_i > R_K \right\} \\ & \geq \Pr \left\{ (r_i - R_K)^{-\alpha} + \dots + (r_i - R_K)^{-\alpha} \leq K\tau N^{-\frac{\alpha}{2}} | r_i > R_K \right\} \\ & = \Pr \left\{ (r_i - R_K)^{-\alpha} \leq \tau N^{-\frac{\alpha}{2}} | r_i > R_K \right\} \end{aligned} \quad (92)$$

Following the steps used to derive (80), we get the following with probability 1.

$$\lim_{N \rightarrow \infty} \Pr \left\{ N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha} + \dots + N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha} \leq K\tau | r_i > R_K \right\} \geq 1 - \frac{\tau^{-2/\alpha}\pi\lambda}{c} 1_{\left\{ \left( \frac{\pi\lambda}{c} \right)^{\alpha/2} < \tau \right\}} \quad (93)$$

Using similar steps we can get

$$\Pr \left\{ N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha} + \dots + N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha} \leq K\tau | r_i > R_K \right\} \leq \Pr \left\{ (r_i + R_K)^{-\alpha} \leq \tau N^{-\frac{\alpha}{2}} | r_i > R_K \right\} \quad (94)$$

Following the steps used to derive (83), we get the following with probability 1.

$$\lim_{N \rightarrow \infty} \Pr \left\{ N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha} + \dots + N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha} \leq K\tau | r_i > R_K \right\} \leq 1 - \frac{\tau^{-2/\alpha}\pi\lambda}{c} 1_{\left\{ \left( \frac{\pi\lambda}{c} \right)^{\alpha/2} < \tau \right\}} \quad (95)$$

Since as  $n, N \rightarrow \infty$ ,  $r_i > R_K$  with probability 1, we conclude the following with probability 1.

$$\lim_{N \rightarrow \infty} \Pr \left\{ \frac{1}{K} \left( N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha} + \dots + N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha} \right) \leq \tau | r_i > R_K \right\} = 1 - \frac{\tau^{-2/\alpha}\pi\lambda}{c} 1_{\left\{ \left( \frac{\pi\lambda}{c} \right)^{\alpha/2} < \tau \right\}} = G(\tau).$$

Following similar steps used to prove (91), we have that the e.d.f. of  $\frac{1}{K} \left( N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha} + \dots + N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha} \right)$  converges with probability 1 to the RHS of the equation above. Since the joint e.d.f. of

$N^{\frac{\alpha}{2}}r_{i,1}^{-\alpha}, \dots, N^{\frac{\alpha}{2}}r_{i,K}^{-\alpha}$  converges with probability 1 to  $H(\tau_1, \dots, \tau_K)$ , the CDF of

$\frac{1}{K}(\tau_1 + \dots + \tau_K)$ , where the  $\tau_i$  terms have joint CDF  $H(\tau_1, \dots, \tau_K)$ , must converge with probability 1 to  $G(\tau)$ .

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